Design & Analysis of Experiments

Orthogonal Latin Squares:

Two latin squares each of the same order, say, r are said to be orthogonal if when one is superimposed on the other each symbol of one falls on each symbol of the other once and only once.

For example; the following two latin squares are orthogonal.

Square I

A	B	C
B	C	A
C	A	B

Square II

α	β	γ
γ	α	β
β	γ	α

Defn: **Latin Squares**

A Latin square of order r is an arrangement of r symbols in r^2 units arranged in an $r \times r$ square such that every symbol occurs ones in each row and once in each column of the square.

A set of latin squares is said to be mutually orthogonal if every pair of these squares is orthogonal.

Maximum Number of Orthogonal Latin Squares:

The number of ortho. latin sq. of order r can be at the most $r-1$.

This can be proved as below.

A L.S is said to be in a substandard form if the symbols occur in its first row in the natural order. If the symbols used are the alphabets A, B, C, etc., then in the first row they occur in the alphabetic order to make a LS substandard. It is always possible to write a ortho. L.S in substandard form by having a suitable permutation over the symbols.

For example, the following LS can be written in the substandard form by permuting D to A, A to B, C to C and B to D.

Standard Square

A	B	C	D
B	C	D	A
D	A	B	C
C	D	A	B

Original Square

D	A	C	B
A	C	B	D
B	D	A	C
C	B	D	A

We assume that all the orthogonal LS are in the substandard form.

In an Ortho. LS of order r any of $r-1$ symbols can be written in the second position of the first column, the remaining symbol occur in the first position of this column. As in each square a different symbol is to occur in this position, the number of such squares can be at the most $r-1$.

When r is either a prime or a prime power all the $(r-1)$ ortho. L.S can be constructed. When r is the product of different primes or prime powers, MacNaish (1922) and Mann (1942) gave methods of construction of $(S-1)$ ortho. squares of order r where S is the minimum prime or prime power factor of r.

We have given below a unified method of construction of ortho. LS of order r, by which $(t-1)$ ortho. squares can be obtained where,

(i) $t = S$ if r is the product of different primes and S is the minimum of the prime (or) prime power factor of r and

(ii) $t = r$ if r is a prime or a prime power.

Construction of ortho. Latin Squares:

Let $(r = s_1, s_2, \ldots, s_p)$ where each factor, s_i is either a prime or a prime power. We shall use

(Galois field) $\Rightarrow$ GF

the s_i elements of $GF(S_i)$ $(i = 1, 2, \ldots, p)$ for forming combinations of elements of the p different fields as below.

Let us combine the elements from the p different fields taking one from each field in all possible ways. There are evidently r such combinations. If r is a prime or prime power, then $p = 1$ and each such combination is just an element of its field. We shall use such combinations of the p field elements as symbols for writing the Latin Squares.

Let the r combinations be written in a row and again in a column so as to obtain the summation table of all possible sums, two by two, of the row-column combination. This column will be called the principal column and the row, the principal row.

By addition or multiplication of two combinations means addition or multiplication of each pair of corresponding elements, in the two combinations in the respective fields.

It can be easily seen that the summation table gives a Latin Square. Next, each combination in the principal column is multiplied by a combination, say, $(a_1, a_2, \ldots, a_p)$, where $a_i \neq 0$ or 1 $(i = 1, 2, \ldots, p)$. The resultant column is the second principal column. Again, another summation table is formed by using this second principal column and the first principal row. This table gives a second latin square which is orthogonal to the one obtained earlier.

Again, a third principal column is obtained by multiplying the different elements in the first principal column by another multiplier, say, $(b_1, b_2, \ldots, b_p)$; where $b_i \neq a_i$ or 1 or 0 $(i = 1, 2, \ldots, p)$ that is, the multipliers are so choosen that no element in any field is repeated in the different multipliers. A third latin square is obtained by adding the third principal column and the first principal row. This square is orthogonal to the previous two. This process is continued till suitable multipliers are available.

If s is a prime or a prime power, each multiplier combination consists of only one element. We can, therefore, get $(r-2)$ multiplier which are the different non zero elements in its field other than unity.

Construction of orth. L.S of order 4:

The elements in G.F (2^2) are $0, 1, \alpha, \alpha^2$ with $\alpha^2+\alpha+1$ as the minimum fn. The three summation table gives three ortho. L.S.

Tables of summation of the elements of G.F (2^2).

First Principal Column	Principal Row			
	0	1	α	α^2
0	0	1	α	α^2
1	1	0	α^2	α
α	α	α^2	0	1
α^2	α^2	α	1	0

Second Principal Column obtained by multiplying by α				
0	0	1	α	α^2
α	α	α^2	0	1
α^2	α^2	α	1	0
1	1	0	α^2	α

Third Principal Column obtained by multiplying by α^2				
0	0	1	α	α^2
α^2	α^2	α	1	0
1	1	0	α^2	α
α	α	α^2	0	1